1. $I = \displaystyle\int_{\pi/4}^{\pi/2} \frac{dt}{\sin t}$

On pose $u = \cos t = \varphi(t)$.

t	u
$\pi/4$	$1/\sqrt{2}$
$\pi/2$	0

φ est C^1 sur $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ donc le chgt de variable est licite.

$du = -\sin(t).dt$

$$I = \int_{\pi/4}^{\pi/2} \frac{-\sin t \, dt}{-\sin^2 t} = \int_{\pi/4}^{\pi/2} \frac{-\sin(t) dt}{\cos^2(t) - 1} = \int_{1/\sqrt{2}}^{0} \frac{du}{u^2 - 1}$$

Rem: on a utilisé la formule de chgt de variable "à l'envers" :

$$\int_{\alpha}^{\beta} f(\varphi(t)) \times \varphi'(t) \, dt = \int_a^b f(u) \, du$$

ce qui nous a évité d'écrire $t = \arccos(u)$.

Donc $I = \displaystyle\int_0^{1/\sqrt{2}} \frac{du}{1 - u^2}$ or $\dfrac{1}{1-u^2} = \dfrac{1}{(1-u)(1+u)} = \dfrac{1}{2}\left(\dfrac{1}{1+u} + \dfrac{1}{1-u}\right)$

avec les techniques de décomposition d'une fraction rationnelle vues p145 chap 5.

Donc $I = \dfrac{1}{2} \times \displaystyle\int_0^{1/\sqrt{2}} \left(\frac{1}{1+u} + \frac{1}{1-u}\right) du$

$$I = \frac{1}{2} \times \Big[\ln(|1+u|) - \ln(|1-u|) \Big]_{u=0}^{u=\frac{1}{\sqrt{2}}}$$

$$I = \frac{1}{2} \left(\ln\left(1 + \frac{1}{\sqrt{2}}\right) - \ln\left(1 - \frac{1}{\sqrt{2}}\right) - 0 + 0 \right)$$

$$I = \frac{1}{2} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$$ Mais $\frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{(\sqrt{2}+1)^2}{2-1} = 3+2\sqrt{2}$

donc $\boxed{I = \frac{1}{2} \ln(3+2\sqrt{2})}$

2. $I = \displaystyle\int_0^{\pi/4} \frac{dx}{1+\cos^2(x)}$

On pose $u = \tan(x) = \varphi(x)$

x	u
0	0
$\pi/4$	1

φ est C^1 sur $\left[0, \frac{\pi}{4}\right]$ donc le chgt de variable est licite.

$$du = (1+\tan^2 x)\,dx = \frac{1}{\cos^2 x}\,dx$$

$$I = \int_0^{\pi/4} \frac{\frac{dx}{\cos^2 x}}{\frac{1}{\cos^2 x} + 1} = \int_0^{\pi/4} \frac{\frac{dx}{\cos^2 x}}{2 + \tan^2 x} = \int_0^1 \frac{du}{2+u^2}$$

or $\frac{1}{2+u^2} = \frac{1}{2} \times \frac{1}{1+\frac{u^2}{2}} = \frac{1}{2} \times \frac{1}{1+\left(\frac{u}{\sqrt{2}}\right)^2} = \frac{1}{\sqrt{2}} \times \frac{\frac{1}{\sqrt{2}}}{1+\left(\frac{u}{\sqrt{2}}\right)^2}$

$$I = \frac{1}{\sqrt{2}} \times \int_0^1 \frac{\frac{1}{\sqrt{2}}}{1+\left(\frac{u}{\sqrt{2}}\right)^2} du = \frac{1}{\sqrt{2}} \left[\arctan\left(\frac{u}{\sqrt{2}}\right)\right]_0^1$$

$$I = \frac{1}{\sqrt{2}} \left(\arctan\left(\frac{1}{\sqrt{2}}\right) - \arctan 0\right) = \boxed{\frac{1}{\sqrt{2}} \arctan\left(\frac{1}{\sqrt{2}}\right)}$$

3. $$I = \int_{1/2}^{2} \cos\left(\frac{x}{1+x^2}\right) \times \frac{\ln x}{x} dx$$

On pose $x = \frac{1}{t} = \varphi(t)$

x	t
2	1/2
1/2	2

φ est C^1 sur $\left[\frac{1}{2}, 2\right]$ donc le chgt de variable est licite.

$dx = -\frac{1}{t^2} dt$

$$I = \int_2^{1/2} \cos\left(\frac{\frac{1}{t}}{1+\frac{1}{t^2}}\right) \times \frac{\ln(1/t)}{1/t} \times \frac{-1}{t^2} dt$$

$$I = \int_2^{1/2} \cos\left(\frac{t}{1+t^2}\right) \times (-t \ln t) \times \frac{-dt}{t^2}$$

$$I = -\int_{1/2}^{2} \cos\left(\frac{t}{1+t^2}\right) \times \frac{\ln t}{t} dt = -I$$

Donc $\boxed{I = 0}$