On a puisque $p \in [\![1, n]\!]$.

$$\mathrm{Vect}(\vec{u_1}, \dots, \vec{u_n}) = \mathrm{Vect}(\vec{u_1}, \dots, \vec{u_p}, \vec{u_{p+1}}, \dots, \vec{u_n})$$
$$\underset{[\text{th}41\ \text{chap}13]}{=} \mathrm{Vect}(\vec{u_1}, \dots, \vec{u_p}) + \mathrm{Vect}(\vec{u_{p+1}}, \dots, \vec{u_n})$$

On a donc :

$$\mathrm{rg}(\vec{u_1}, \dots, \vec{u_n}) \overset{\text{def}}{=} \dim\left(\mathrm{Vect}(\vec{u_1}, \dots, \vec{u_n})\right)$$
$$= \dim\left(\mathrm{Vect}(\vec{u_1}, \dots, \vec{u_p}) + \mathrm{Vect}(\vec{u_{p+1}}, \dots, \vec{u_n})\right)$$
$$\underset{[\text{cor}20\ \text{chap}15]}{\leqslant} \dim\left(\mathrm{Vect}(\vec{u_1}, \dots, \vec{u_p})\right) + \dim\left(\mathrm{Vect}(\vec{u_{p+1}}, \dots, \vec{u_n})\right)$$

Donc $\mathrm{rg}(\vec{u_1}, \dots, \vec{u_n}) \leqslant \mathrm{rg}(\vec{u_1}, \dots, \vec{u_p}) + \mathrm{rg}(\vec{u_{p+1}}, \dots, \vec{u_n})$

D'autre part : $\mathrm{Vect}(\vec{u_{p+1}}, \dots, \vec{u_n})$ est engendré par les $n-(p+1)+1 = n-p$ vecteurs $(\vec{u_{p+1}}, \dots, \vec{u_n})$

donc $\dim\left(\mathrm{Vect}(\vec{u_{p+1}}, \dots, \vec{u_n})\right) \leqslant n-p$ [th 8 chap 15]

Ainsi $\boxed{\mathrm{rg}(\vec{u_1}, \dots, \vec{u_n}) \leqslant \mathrm{rg}(\vec{u_1}, \dots, \vec{u_p}) + n - p}$