1. On fixe $n \in \mathbb{N}^*$ et $p \in \mathbb{N}$.

$$I_{n,p} = \int_{-1}^{1} \underbrace{(1+x)^n}_{u(x)} \underbrace{(1-x)^p}_{v'(x)} \, dx = \left[-(1+x)^n \frac{(1-x)^{p+1}}{p+1} \right]_{-1}^{1} + \frac{n}{p+1} \int_{-1}^{1} (1+x)^{n-1} (1-x)^{p+1} \, dx$$

$$\begin{cases} u'(x) = n(1+x)^{n-1} \\ v(x) = -\dfrac{(1-x)^{p+1}}{p+1} \end{cases}$$

$$= 0 - 0 + \frac{n}{p+1} I_{n-1,p+1}$$

$$= \frac{n}{p+1} \times I_{n-1,p+1}$$

On a donc :

$$I_{n,p} = -\frac{n}{p+1} I_{n-1,p+1} = \left(\frac{n}{p+1}\right)\left(\frac{n-1}{p+2}\right) I_{n-2,p+2}$$

$$= \left(\frac{n}{p+1}\right)\left(\frac{n-1}{p+2}\right)\left(\frac{n-2}{p+3}\right) I_{n-3,p+3} = \cdots\cdots$$

$$= \left(\frac{n}{p+1}\right) \times \left(\frac{n-1}{p+2}\right) \times \cdots \times \left(\frac{1}{p+n}\right) \times I_{0,p+n}$$

$$= \frac{n! \times p!}{(p+n)!} \times I_{0,p+n}$$

Or $I_{0,p+n} = \displaystyle\int_{-1}^{1} (1-x)^{p+n} \, dx = \left[-\frac{(1-x)^{p+n+1}}{p+n+1} \right]_{-1}^{1} = \frac{2^{p+n+1}}{p+n+1}$

Donc $\boxed{\forall (n,p) \in \mathbb{N}^2, \; I_{n,p} = \dfrac{n! \times p!}{(n+p+1)!} \times 2^{n+p+1}}$

2. Soit $(n,p) \in \mathbb{N}^2$.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

$$(1-x)^p = \sum_{k=0}^{p} \binom{p}{k} (-1)^k x^k$$

donc $$(1+x)^n (1-x)^p = \sum_{k=0}^{n} \left(\sum_{j=0}^{p} \binom{n}{k}\binom{p}{j} (-1)^j x^{k+j} \right)$$

Par linéarité de l'intégrale :

$$I_{n,p} = \sum_{k=0}^{n} \left(\sum_{j=0}^{p} \binom{n}{k}\binom{p}{j} (-1)^j \times \int_{-1}^{1} x^{k+j}\, dx \right)$$

or $$\int_{-1}^{1} x^{k+j}\, dx = \left[\frac{x^{k+j+1}}{k+j+1} \right]_{x=-1}^{x=1} = \frac{1-(-1)^{k+j+1}}{k+j+1}$$

donc $$\boxed{I_{n,p} = \sum_{k=0}^{n} \left(\sum_{j=0}^{p} \binom{n}{k} \times \binom{p}{j} \times \frac{(-1)^j - (-1)^{k+1}}{k+j+1} \right)}$$

Rem : On préfère sincèrement la formule du 1.

3. $W_{2n+1} = \int_0^{\pi} \sin^{2n+1}(t)\,dt$

On pose $x = \cos t = \varphi(t)$

x	t
1	0
-1	π

φ est C^1 sur $[0,\pi]$ donc le changement de variable est licite.

$dx = -\sin t\,dt$

$$W_{2n+1} = -\int_0^{\pi} (1-\cos^2 t)^n \times (-\sin t)\,dt$$

$$= -\int_1^{-1} (1-x^2)^n\,dx = \int_{-1}^{1} (1+x)^n(1-x)^n\,dx$$

$$= I_{n,n}$$

donc $\boxed{W_{2n+1} = \frac{(n!)^2}{(2n+1)!} \times 2^{2n+1}}$