TD2 ①

1. • $\displaystyle\sum_{k=1}^{n} \ln\left(1+\frac{1}{k}\right) = \sum_{k=1}^{n}\left(\ln(k+1) - \ln k\right) \underset{\text{télescop.}}{=} \ln(n+1) - \ln 1$

$= \boxed{\ln(n+1)}$

• $\displaystyle\sum_{k=3}^{n} \ln\left(1-\frac{2}{k}\right) = \sum_{k=3}^{n}\left(\ln(k-2) - \ln k\right)$

$\displaystyle \underset{\text{linéarité}}{=} \sum_{k=3}^{n}\ln(k-2) - \sum_{k=3}^{n}\ln(k)$

$\displaystyle \underset{k'=k-2}{=} \sum_{k'=1}^{n-2}\ln(k') - \sum_{k=3}^{n}\ln(k)$

$\displaystyle = \ln 1 + \ln 2 + \cancel{\sum_{k=3}^{n-2}\ln(k)} - \cancel{\sum_{k=3}^{n-2}\ln(k)} - \ln n - \ln(n-1)$

k, k' variables muettes

$= \boxed{\ln\left(\frac{2}{n(n-1)}\right)}$

autre méthode :

$\displaystyle\sum_{k=3}^{n} \ln\left(1-\frac{2}{k}\right) = \sum_{k=3}^{n}\left(\ln(k-2) - \ln(k-1) + \ln(k-1) - \ln k\right)$

$\displaystyle \underset{\text{linéarité}}{=} \sum_{k=3}^{n}\left(\ln(k-2) - \ln(k-1)\right) + \sum_{k=3}^{n}\left(\ln(k-1) - \ln k\right)$

$\underset{\text{télescop.}}{=} \ln 1 - \ln(n-1) + \ln 2 - \ln n$

$= \ln\left(\frac{2}{n(n-1)}\right)$

②

- $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n+1} = \boxed{\frac{n}{n+1}}$

car $\frac{1}{k(k+1)} = \frac{k+1-k}{k(k+1)} = \frac{k+1}{k(k+1)} - \frac{k}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$

- $\prod_{k=1}^{n} \frac{k}{k+1} = \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{4}} \times \dots \times \frac{\cancel{n}}{n+1} = \boxed{\frac{1}{n+1}}$

- $\prod_{k=1}^{n} e^{k^2} = e^{1^2} \times e^{2^2} \times \dots \times e^{n^2} = e^{1^2+2^2+\dots+n^2} = \boxed{e^{\frac{n(n+1)(2n+1)}{6}}}$

- $\sum_{k=2}^{n} \ln\left(\frac{k^2-1}{k^2}\right) = \sum_{k=2}^{n}\left(\ln(k+1) + \ln(k-1) - 2\ln k\right)$

$\overset{\text{linéarité}}{=} \sum_{k=2}^{n} \ln(k+1) + \sum_{k=2}^{n} \ln(k-1) - 2\sum_{k=2}^{n} \ln k$

$= \sum_{k'=3}^{n+1} \ln(k') + \sum_{k''=1}^{n-1} \ln(k'') - 2 \cdot \sum_{k=2}^{n} \ln k$

$k' = k+1$
$k'' = k-1$

$= \ln(n+1) - \ln 2 + \sum_{k=2}^{n} \ln(k) + \ln 1 - \ln n + \sum_{k=2}^{n} \ln k$

$\quad - 2 \times \sum_{k=2}^{n} \ln k$

k', k'', k variables muettes

$= \boxed{\ln\left(\frac{n+1}{2n}\right)}$