TD 2

1. $$\binom{n}{i}\times\binom{n-i}{p-i} = \frac{n!}{i!\times(n-i)!}\times\frac{(n-i)!}{(p-i)!\times(n-p)!} = \frac{n!}{i!\times(p-i)!\times(n-p)!}$$

si on multiplie par $1=\frac{p!}{p!}$ on trouve :

$$\binom{n}{i}\times\binom{n-i}{p-i}=\binom{p}{i}\times\binom{n}{p}$$

Donc $$\sum_{i=0}^{p}\binom{n}{i}\times\binom{n-i}{p-i} = \sum_{i=0}^{p}\binom{p}{i}\times\binom{n}{p} = \binom{n}{p}\times\sum_{i=0}^{p}\binom{p}{i} = \binom{n}{p}\times 2^p$$

2. $$\sum_{k=0}^{n}\binom{p+k}{k} = \binom{p}{0}+\binom{p+1}{1}+\binom{p+2}{2}+\cdots+\binom{p+n}{n}$$

Or $\binom{p}{0}=1=\binom{p+1}{0}$. Donc :

$$\binom{p}{0}\quad \binom{p+1}{1}\quad \binom{p+2}{2}\quad \cdots\quad \binom{p+n}{n}$$

$$\sum_{k=0}^{n}\binom{p+k}{k} = \binom{p+1}{0}+\binom{p+1}{1}+\binom{p+2}{2}+\cdots+\binom{p+n}{n}$$

$$\overset{\text{Pascal}}{=} \binom{p+2}{1}+\binom{p+2}{2}+\cdots+\binom{p+n}{n}$$

$$\overset{\text{Pascal}}{=} \binom{p+2}{2}+\cdots+\binom{p+n}{n}$$

$$=\cdots=\binom{p+n}{n-1}+\binom{p+n}{n}=\binom{p+n+1}{n}$$

On a additionné n termes diagonaux dans le triangle de Pascal.

3. D'après la formule de Pascal :

$$\binom{k}{n} = \binom{k+1}{n+1} - \binom{k}{n+1}$$

Donc par télescopage : $\displaystyle\sum_{k=n}^{p} \binom{k}{n} = \binom{p+1}{n+1} - \binom{n}{n+1} = \binom{p+1}{n+1}$

$$\binom{n}{n}$$

$$\binom{n+1}{n}$$

$$\vdots$$

$$\binom{p}{n}$$

On a additionné $n-p+1$ termes en colonne dans le triangle de Pascal.

4. On pose $\displaystyle S_n = \sum_{p=0}^{n} \left(\sum_{k=p}^{n} 3^{n-k} \times \binom{n}{k} \times \binom{k}{p} \right)$

D'après Fubini $\displaystyle S_n = \sum_{k=0}^{n} \left(\sum_{p=0}^{k} 3^{n-k} \binom{n}{k} \binom{k}{p} \right)$

Pour $k \in [\![0,n]\!]$: $\displaystyle\sum_{p=0}^{k} 3^{n-k} \cdot \binom{n}{k} \cdot \binom{k}{p} = \binom{n}{k} 3^{n-k} \times \sum_{p=0}^{k} \binom{k}{p}$

$$= \binom{n}{k} \cdot 3^{n-k} \cdot 2^k$$

Donc $\displaystyle S_n = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} \cdot 2^k = (3+2)^n = 5^n$.

vérification : pour $n=0$, $S_n = 3^0 \times \binom{0}{0} \times \binom{0}{0} = 1$

et $5^n = 1$ ok.